

NR 062-160

IMM-NYU 328
FEBRUARY 1965



NEW YORK UNIVERSITY
COURANT INSTITUTE OF
MATHEMATICAL SCIENCES

EXISTENCE, UNIQUENESS, AND STABILITY FOR
NONLINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS
IN THE NEUTRAL CASE

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PREPARED UNDER
CONTRACT NO. NONR-285(55)
OFFICE OF NAVAL RESEARCH

IMM-328
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C. 1

Introduction

It is well known that a linear homogeneous ordinary differential equation with constant coefficients, whose characteristic roots lie in the left half plane, possesses only solutions that decay exponentially and all solutions of a slight nonlinear perturbation of the differential equation also have this property. This result is due to Liapounov (see Coddington and Levinson [3]). Bellman [1] showed a similar result for the differential-difference equation

$$(0.1) \quad \sum_{\mu=0}^m \sum_{\nu=0}^n a_{\mu\nu} u^{(\nu)}(t+b_{\mu}) = 0$$

where $0 = b_0 < b_1 < \dots < b_m$.

Specifically, if all roots s of its characteristic equation

$$(0.2) \quad \sum_{\mu=0}^m \sum_{\nu=0}^n a_{\mu\nu} s^{\nu} \exp(b_{\mu} s) = 0$$

satisfy $\Re(s) \leq b < 0$, then all solutions of the differential-difference equation and all solutions of a slight nonlinear perturbation thereof decay exponentially as $t \rightarrow +\infty$.

What happens if all roots s of the characteristic equation (0.2) lie in the left half plane, but are asymptotic to the imaginary axis? This question arose in a study of a transmission line problem made by Miranker

[9]. This point will be discussed in the appendix. The primary reason, however, for studying this problem is that the rate of decay of solutions depends upon their smoothness.

Using the fact established by Wright [14] that every solution of (0.1) can be represented as an infinite sum of exponential solutions e^{st} (where s are solutions of (0.2)) for $t \geq t_0$ provided $a_{mn} \neq 0$, Hahn [8] demonstrated that all sufficiently smooth solutions of

$$(0.3) \quad \sum_{\nu=0}^n [a_{\nu} u^{(\nu)}(t+\delta) + b_{\nu} u^{(\nu)}(t)] = 0$$

decay at least as fast as some negative power of t as $t \rightarrow +\infty$ if $\operatorname{Re}(s) < 0$ for all roots s of

$$(0.4) \quad \sum_{\nu=0}^n (a_{\nu} e^{\delta s} + b_{\nu}) s^{\nu} = 0.$$

He also showed that under certain conditions there exist solutions which decay no faster than some power of t . The nonlinear theory and even the inhomogeneous linear theory, however, have not as yet been studied. It is the primary aim of this paper to tackle this problem.

We wish first to give some background information on this problem. Langer, in an expository article [3], dealt with the characteristic equation (0.2). He found that there are infinitely many solutions thereof. These roots can be grouped into chains, along each of which $\operatorname{Im}(s) \rightarrow \pm \infty$. If

$\Re \lambda(s) \rightarrow -\infty$ along the chain, we have a retarded chain; if $\Re \lambda(s)$ is bounded, we have a neutral chain; and if $\Re \lambda(s) \rightarrow +\infty$, we have an advanced chain. He then gave criteria for determining what type chains (0.2) has. In particular, if $a_{mn} \neq 0$, we have no advanced chains; if $a_{mn} \neq 0$ and $a_{on} \neq 0$, we have a neutral chain only; if $a_{mn} \neq 0$ and $a_{on} = 0$, there is at least one retarded chain; and if $a_{mn} = 0$ and $a_{on} \neq 0$, there are no retarded chains and at least one advanced chain. Cooke [4] handled this problem for systems of differential-difference equations. Pontryagin [10] gave necessary and sufficient conditions for the negativity of the real parts of all zeros of (0.2) where the b_μ are commensurable.

Considering (0.1) with data prescribed on $[0, b_m]$ or (0.3) with data prescribed on $[0, \delta]$, we note that if the associated characteristic equation has advanced chains only, then all solutions lose smoothness as t increases for we can solve for $u^{(i)}(t + b_m)$ where $i < n$ in terms of higher derivatives in the past; if the associated characteristic equation has retarded chains only, then all solutions gain smoothness as t increases; and if the associated characteristic equation has neutral chains only, then all solutions are as smooth as they were initially (Bellman and Cooke [2]). Also, (0.4) has exactly one chain and if we require $\Re \lambda(s) < 0$ but there exists no $b < 0$ such that $\Re \lambda(s) \leq b$, then it is obviously neutral.

The preceding paragraph leads one to conjecture that smoothness plays a role. In addition it would seem reasonable that the rate of approach

of the characteristic roots to the imaginary axis should enter the discussion. We shall show this to be true and shall give a precise qualitative formulation of these facts.

Chapters I, II, and III will be devoted primarily to a discussion of

$$(0.5) \quad u'(t+\delta) - u'(t) + au(t+\delta) = g(t, u(t), u(t+\delta)), \quad a > 0, \delta > 0.$$

We remark parenthetically that it is to be expected that the corresponding characteristic roots hug the imaginary axis for if we consider the linear part

$$(0.6) \quad u'(t+\delta) - u'(t) + au(t+\delta) = 0$$

and divide by δ and let δ get small we obtain approximately $u''(t) + \frac{a}{\delta} u(t) = 0$ whose characteristic roots lie on the imaginary axis.

In Chapter I we consider (0.6) and show it is of desired type. Then utilizing the expansion theorem of Wright, we prove that any solution of (0.6) which has $(m+1)$ continuous derivatives with its $(m+1)$ st derivative satisfying a Holder condition with Holder exponent ϵ and $m+\epsilon > 0$ decays at least as fast as $t^{-(m+\epsilon)/2}$. This result was obtained independently of Hahn's, our estimate for decay is slightly sharper than his, we obtain decay results for the derivatives of $u(t)$, and we make clear the relationship between smoothness and decay. Then we give an example showing that we have no

better than decay as some power of t . Then we give an example of a differential-difference equation all of whose characteristic roots s satisfy $\operatorname{Re}(s) < 0$, yet there is a solution of the linear homogeneous equation which is unbounded as $t \rightarrow +\infty$.

The analog of Wright's expansion is derived for the inhomogeneous equation

$$(0.7) \quad u'(t+\delta) - u'(t) + au(t+\delta) = f(t)$$

in Chapter II and then applied to obtain decay results for this case. Without proof, we then state the corresponding decay results for the more general equation

$$(0.8) \quad \sum_{\nu=0}^n [a_{\nu} u^{(\nu)}(t+\delta) + b_{\nu} u^{(\nu)}(t)] = f(t).$$

Specifically, if $f(t)$ decays sufficiently fast and is sufficiently smooth, then $u(t)$ decays at least as fast as $t^{-(n-m+1)/\epsilon}$ if $u(t)$ has $m+n$ continuous derivatives initially and the characteristic roots of (0.4) approach the imaginary axis approximately as $-|r|^{-\epsilon}$ where s_r are the roots.

Then in Chapter III we discuss the nonlinear equation (0.5). Applying the Cauchy-Peano existence theorem for ordinary differential equations and the results of Chapter II, we find that if m is an integer ≥ 3 , $g(t, u, v)$ and all its partial derivatives up to order m are continuous, $g(t, u, v) = O(u^2 + v^2)$,

and the initial data is small and has at least $m+1$ continuous derivatives all of which are sufficiently small, then we have the solution $u(t)$ decaying at least as fast as $t^{-m/2}$.

In the appendix we show how the results of Chapter III enable us to prove stability for the wave equation with a certain nonlinear interface condition. We wish to note that Hoheisel [7] pointed out how Laplace's equation with certain boundary conditions can lead to differential-difference equations.

I wish to thank now Professor Jürgen K. Moser for suggesting the problem and giving advice and encouragement for its solution.

Chapter I

The Homogeneous Case

It will be the primary purpose of this chapter to show that the rate of decay of solutions of

$$(1.0.1) \quad u'(t+\delta) - u'(t) + au(t+\delta) = 0, \quad a > 0, \delta > 0$$

depends upon the smoothness of these solutions.

In the first section we shall prove some basic lemmas, among them the distribution of s such that e^{st} is a solution of (1.0.1). In section two we shall express all solutions of (1.0.1) in terms of initial data assigned on $0 \leq t \leq \delta$ and the aforementioned exponentials. Here we shall delete much of the proof because it is discussed in detail for (0.1) in Wright [14] and again in detail for the inhomogeneous equation (0.7) in Chapter II. Section three will be devoted to the main result of this chapter, a solution which has an $(m+1)$ st derivative which satisfies a Holder condition with exponent ϵ decays at least as fast as $t^{(-m-\epsilon)/2}$. An example showing that there exists a solution which (has $m+1$ continuous derivatives and) decays at best as $t^{(-m-1-\mu)/2}$ (μ arbitrary > 0) constitutes section four. Section five uses the result of section four to give an example of a differential-difference equation all of whose solutions of the form e^{st} have the property $\operatorname{Re}(s) < 0$, yet there exists a bona fide solution which becomes unbounded as $t \rightarrow +\infty$.

1. Some Basic Lemmas

We wish first to make an elementary observation regarding (1.0.1).

If we are given $u(0)$ and $u'(t)$ continuous on $0 \leq t \leq \delta$, then there exists a unique solution of (1.0.1) for all $t \geq 0$ and $u'(t)$ is continuous, except possibly at points of the form $n\delta$, n being a positive integer. This can be simply seen by proceeding interval by interval, each interval of length δ , at each step solving the ordinary differential equation resulting from treating $u'(t)$, which is known, as the inhomogeneous term. The discontinuities of $u'(t)$ at $n\delta$ can be remedied by simply requiring continuity at δ . To avoid this difficulty, we shall from now on understand continuity on the initial interval to include continuity from the right at the right endpoint.

Now we shall formulate this remark for higher order derivatives as Lemma 1.1.1. It too follows by the interval by interval procedure.

LEMMA 1.1.1. If for (1.0.1), $u(t)$ is given on $0 \leq t \leq \delta$ such that $u^{(m+1)}(t)$ satisfies a Holder condition with exponent ϵ there, then there exists a unique solution $u(t)$ for all $t \geq 0$, $u(t)$ has $m+1$ continuous derivatives for all $t \geq 0$, and the $(m+1)^{\text{st}}$ derivative satisfies a Holder condition with exponent ϵ on any finite interval.

Next we shall state a lemma which will permit us to use the Laplace transform which is needed to derive the expansion theorem in Section two. We omit the proof for it can be found in Wright [13].

LEMMA 1.1.2. If for (1.0.1) we are given $u'(t)$ is continuous on $0 \leq t \leq \delta$, then $|u(t)|, |u'(t)| \leq C_1 \exp(C_2 t)$.

DEFINITION 1.1.1. Suppose e^{st} is a solution of (1.0.1). Then s satisfies

$$(1.1.1) \quad \tau(s) = se^{\delta s} - s + ae^{\delta s} = 0.$$

(1.1.1) is called the characteristic equation of (1.0.1).

In the next three lemmas we shall be concerned with the distribution of roots of (1.1.1).

LEMMA 1.1.3. All roots s of (1.1.1) have $\operatorname{Re}(s) < 0$.

Proof: Transposing s and dividing by $se^{\delta s}$ in (1.1.1) yields

$$(1.1.2) \quad e^{-\delta s} = 1 + \frac{a}{s}.$$

Now, $\operatorname{Re}(s) \geq 0$ implies $|e^{-\delta s}| \leq 1$. This in turn implies

$$(1.1.3) \quad (e^{-\delta s}) \begin{cases} < 1 \text{ for } s \neq \frac{2\pi ni}{\delta}, \text{ } n \text{ integer} \\ = 1 \text{ for } s = \frac{2\pi ni}{\delta}, \text{ } n \text{ integer.} \end{cases}$$

Also, we know that if $\operatorname{Re}(s) \geq 0$, then

$$(1.1.4) \quad \operatorname{Re}\left(1 + \frac{a}{s}\right) = 1 + a \operatorname{Re}\left(\frac{s}{|s|^2}\right) \begin{cases} > 1 \text{ for } s \neq \lambda i \\ = 1 \text{ for } s = \lambda i \neq 0. \end{cases}$$

Combining (1.1.2), (1.1.3), and (1.1.4), we obtain that the only possible solutions of (1.1.1) such that $\operatorname{Re}(s) \geq 0$ are $s = \frac{2\pi ni}{\delta}$. Substitution of

$\frac{2\pi ni}{\delta}$ for s in (1.1.1) yields $a = 0$, but $a \neq 0$.

LEMMA 1.1.4. (1.1.1) has no multiple roots.

Proof: Let

$$(1.1.5) \quad \delta s = p, \delta a = b.$$

Then (1.1.1) becomes

$$(1.1.6) \quad pe^p - p + be^p = 0.$$

For a multiple root, we differentiate and obtain $pe^p + e^p - 1 + be^p = 0$.

Elimination of e^p yields $p^2 + bp - b = 0$. This has exactly two solutions, both of which can be shown not to be solutions of (1.1.6).

LEMMA 1.1.5. For every sufficiently large (positive or negative) integer r , there exists a unique solution s_r of (1.1.1) such that

$$(1.1.7) \quad \left| \operatorname{Re}(s_r) + \frac{(a\delta)^2}{8\pi^2 r^2 \delta} \right| \leq \frac{c_3}{r^4}$$

and

$$(1.1.8) \quad \left| \operatorname{Im}(s_r) - \frac{1}{\delta} \left(2\pi r + \frac{a\delta}{2\pi r} \right) \right| \leq \frac{c_4}{r^3}.$$

Furthermore, all sufficiently large (in absolute value) solutions of

(1.1.1) satisfy these relationships for some r .

(See Figure 1 below .)

$s_{r+1} \cdot$	$\frac{2\pi(r+1)}{\delta}$
$s_r \cdot$	$\frac{2\pi r}{\delta}$
$s_{r-1} \cdot$	$\frac{2\pi(r-1)}{\delta}$
$\cdot$	
$\cdot$	$\frac{2\pi(-r+1)}{\delta}$
$s_{-r+1} \cdot$	$\frac{2\pi(-r)}{\delta}$
$s_{-r} \cdot$	$\frac{2\pi(-r-1)}{\delta}$
$s_{-r-1} \cdot$	

Figure 1 .

Proof: By elementary algebra, (1.1.3) becomes $e^p = 1 - \frac{be^p}{p}$.

Taking logarithms, we obtain

$$(1.1.9) \quad p = \log(1 - \frac{be^p}{p}) = 2\pi ri - \frac{be^p}{p} + \dots$$

Thus, $|p - 2\pi ri| \leq \left| \frac{be^p}{p} \right| + \dots \leq \frac{2b}{|p|}$ for $|p|$ sufficiently large.

Hence,

$$(1.1.10) \quad |p - 2\pi ri| \leq \frac{2b}{\pi r}$$

Note that (1.1.9) is valid for all r . Consider large r and let us look for a p corresponding to r . By convention, in (1.1.9),

$\mathcal{O}_\lambda(\log(1 - \frac{be^p}{p})) = 2\pi r + \lambda$, where $|\lambda| \leq \pi$. Now, p is sufficiently large to apply the result of the preceding paragraph.

Let

$$(1.1.11) \quad p = d + (2\pi r + \epsilon)i$$

in (1.1.6). This yields

$$[d + (2\pi r + \epsilon)i][e^{d_1}(\cos \epsilon + i \sin \epsilon) - 1] + be^{d_1}(\cos \epsilon + i \sin \epsilon) = 0.$$

Breaking this up into real and imaginary parts, we get

$$(1.1.12) \quad de^{d_1} \cos \epsilon - d - (2\pi r + \epsilon)e^{d_1} \sin \epsilon + be^{d_1} \cos \epsilon = 0,$$

$$(1.1.13) \quad de^{d_1} \sin \epsilon + (2\pi r + \epsilon)(e^{d_1} \cos \epsilon - 1) + be^{d_1} \sin \epsilon = 0.$$

Elimination of $2\pi r + \epsilon$ from (1.1.12) and (1.1.13) yields

$$(1.1.14) \quad b(e^{d_1} - \cos \epsilon) = d(2\cos \epsilon - e^{d_1} - e^{-d_1}).$$

In (1.1.14) we expand all terms in their McLaurin expansion and obtain

$$d + \frac{\epsilon^2}{2} = O(d^2) + O(\epsilon^4) + O(d\epsilon^2). \text{ Thus, } d = O(\epsilon^2). \text{ Consequently,}$$

$$d + \frac{\epsilon^2}{2} = O(\epsilon^4). \text{ Substitution of this in (1.1.13) and expansion yields}$$

$$2\pi r(d - \frac{\epsilon^2}{2} + O(\epsilon^4)) + b\epsilon = O(\epsilon^3). \text{ Together with (1.1.10), these}$$

facts imply

$$(1.1.15) \quad 2\pi r(d - \frac{\epsilon^2}{2}) + b\epsilon = O(\frac{1}{r}) ,$$

$$(1.1.16) \quad 2\pi r(d + \frac{\epsilon^2}{2}) = O(\frac{1}{r}) .$$

Then, by simple algebraic manipulations, we obtain

$$(1.1.17) \quad d = \frac{-b^2}{8\pi^2 r^2} + O(\frac{1}{r^{\frac{1}{2}}})$$

and

$$(1.1.18) \quad \epsilon = \frac{b}{2\pi r} + O(\frac{1}{r}) .$$

Combined with (1.1.5) and (1.1.11), these are (1.1.7) and (1.1.3) .

Suppose that corresponding to a given large r , there exist s_r and S_r solutions of (1.1.1) satisfying (1.1.7) and (1.1.3) . Let $s = x+iy$, $\tau(s) = u + iv$. Then, someplace along the line connecting s_r and S_r , the directional derivative of u along this line ($u_x \cos \alpha + u_y \sin \alpha$) equals zero by Rolle's theorem and similarly, someplace along this line, the directional derivative of v along this line ($v_x \cos \alpha + v_y \sin \alpha$) equals zero. But $\Im m(\tau'(s)) (= -u_y = v_x)$ behaves like r in the vicinity of s_r and S_r while $\Re e(\tau'(s)) (= u_x = -v_y)$ is bounded in this neighborhood, independently of r .

The next two lemmas will provide us with estimates of exponential sums which we shall use in Sections 3 of this chapter and Chapter II to obtain decay as $t \rightarrow +\infty$ and in Section 4 of this chapter

to give an example of a solution of (1.1.1) decaying at most as some power of t .

LEMMA 1.1.6. For $t \geq \delta$, $Q > 0$, $\epsilon > 0$, there exist

$C_5(Q, \epsilon) > 0$, $C_6(Q, \epsilon) > 0$ such that

$$(1.1.19) \quad C_5 t^{-\epsilon/2} \leq \sum_{r=1}^{\infty} \frac{\exp(-Qt/r^2)}{r^{1+\epsilon}} \leq C_6 t^{-\epsilon/2}.$$

$$\text{Proof: Let } f(t, x) = \frac{\exp(-Qt/x^2)}{x^{1+\epsilon}}.$$

By differentiating, we find that $f_x(t, x) = 0$ implies $x^2 = \frac{2Qt}{1+\epsilon}$.

Also, $f(t, x)$ is monotonic increasing in x for $x^2 < \frac{2Qt}{1+\epsilon}$ and monotonic decreasing in x for $x^2 > \frac{2Qt}{1+\epsilon}$. Consequently,

$$\begin{aligned} & \int_0^{\infty} \frac{\exp(-Qt/x^2)}{x^{1+\epsilon}} dx + \left[\frac{\exp(-Qt/x^2)}{x^{1+\epsilon}} \right]_{x^2} = \frac{2Qt}{1+\epsilon} \\ (1.1.20) \quad & \geq \sum_{r=1}^{\infty} \frac{\exp(-Qt/r^2)}{r^{1+\epsilon}} \\ & \geq \int_0^{\infty} \frac{\exp(-Qt/x^2)}{x^{1+\epsilon}} dx - \left[\frac{\exp(-Qt/x^2)}{x^{1+\epsilon}} \right]_{x^2} = \frac{2Qt}{1+\epsilon}. \end{aligned}$$

Now,

$$(1.1.21) \quad \left[\frac{\exp(-Qt/x^2)}{x^{1+\epsilon}} \right]_{x^2} = \frac{2Qt}{1+\epsilon} = C_7(Q, \epsilon) t^{-(\frac{1+\epsilon}{2})}.$$

Next we evaluate the integral. Let $y = \frac{\sqrt{Qt}}{x}$. Then

$$(1.1.22) \quad \int_0^{\infty} \frac{\exp(-Qt/x^2)}{x^{1+\varepsilon}} dx = C_3(Q, \varepsilon) t^{-\varepsilon/2}.$$

Thus, for $t \geq C_3(Q, \varepsilon)$, we are done. For $C_3 \geq t \geq \delta$, the result is obvious.

DEFINITION 1.1.2. $\sum_{s_r}$ is to denote the sum over all roots of (1.1.1), the summation being performed in order of ascending absolute value of the roots. We use this notation for simplicity, even though the first few roots may not correspond to some r , because the first few roots are finitely many, simple, and have negative real part and hence can be ignored in all our calculations.

LEMMA 1.1.7. For $t \geq 0$,

$$(1.1.23) \quad \left| \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \right| \leq C_{10}(a, \delta) \log\left(2 + \frac{at}{2\pi}\right).$$

$$\begin{aligned} \text{Proof: } \left| \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \right| &= \left| \sum_{s_r} \frac{\exp(s_r t)}{\exp(\delta s_r) - 1 + \delta s_r} \right| \\ &= \left| \sum_{\psi_r(s_r) > 0} \left(\frac{\exp(s_r t)}{\exp(\delta s_r) - 1 + \delta s_r} \right. \right. \\ &\quad \left. \left. + \frac{\exp(\bar{s}_r t)}{\exp(\delta \bar{s}_r) - 1 + \delta \bar{s}_r} \right) \right| \end{aligned}$$

$$\leq C_{11}(a, \delta) \left[1 + \left| \sum_{r=1}^{\infty} \frac{\exp\left[\left(\frac{-(a\delta)^2}{2r^2\delta} + O\left(\frac{1}{r^4}\right)\right)t\right] \sin\left[\left(\frac{2\pi r}{\delta} + \frac{a}{2\pi r} + O\left(\frac{1}{r^3}\right)\right)t\right]}{r} \right| \right]$$

(by Lemmas 1.1.3 and 1.1.5 and some elementary algebra)

$$\leq C_{12}(a, \delta) \left[1 + \sum_{r \leq \left(\frac{at}{2\pi}\right)^2} \frac{1}{r} + \left| \sum_{r \geq \left(\frac{at}{2\pi}\right)^2} \frac{\sin\left[\left(\frac{2\pi r}{\delta} + \frac{a}{2\pi r}\right)t\right]}{r} \right| \right]$$

$$\leq C_{13}(a, \delta) \left[\log\left(2 + \frac{at}{2\pi}\right) + \left| \sum_{r \geq \left(\frac{at}{2\pi}\right)^2} \frac{\sin\left(\frac{2\pi rt}{\delta}\right)}{r} \right| \right]$$

$$\leq C_{13}(a, \delta) \log\left(2 + \frac{at}{2\pi}\right).$$

2. Solution in Terms of Initial Data

THEOREM 1.2.1. If $u'(t)$ is continuous for $0 \leq t \leq \delta$, then the solution of (1.0.1) for $t \geq \delta$ is

$$(1.2.1) \quad u(t) = \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \left[\frac{-au(\delta)}{s_r} + \int_0^\delta u'(x) \exp(-s_r x) \delta x \right].$$

Proof: See Wright [14].

THEOREM 1.2.2. If $u^{(m+1)}(t)$ is continuous for $0 \leq t \leq \delta$, then the solution of (1.0.1) for $t \geq \delta$ is

$$(1.2.2) \quad u(t) = \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \left[\frac{-au^{(m)}(\delta)}{s_r^{m+1}} + \frac{1}{s_r^m} \int_0^\delta u^{(m+1)}(x) \exp(-s_r x) dx \right].$$

Proof: By integration by parts,

$$\begin{aligned} \int_0^\delta u'(x) \exp(-s_r x) dx &= \frac{-u'(\delta) \exp(-\delta s_r)}{s_r} + \frac{u'(0)}{s_r} \\ &+ \frac{1}{s_r} \int_0^\delta u''(x) \exp(-s_r x) dx. \end{aligned}$$

By (1.0.1) and (1.1.1), $-u'(\delta) \exp(-\delta s_r) + u'(0) - au(\delta) = \frac{-au'(\delta)}{s_r}$.

Thus, $u(t) = \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \left[\frac{-au'(\delta)}{s_r^2} + \frac{1}{s_r} \int_0^\delta u''(x) \exp(-s_r x) dx \right]$. Etcetera.

3. Smoothness and Decay

THEOREM 1.3.1. If $u^{(m+1)}(t)$ is continuous for $0 \leq t \leq \delta$, $u^{(m+1)}(t)$ satisfies a Holder condition with exponent ϵ and constant C_1 ; for $0 \leq t \leq \delta$, and $m+\epsilon > 0$, then the solution of (1.0.1) satisfies the inequality

$$(1.3.1) \quad |u(t)| \leq C_{10}(a, \delta, m+\epsilon) \left\{ \max_{0 \leq x \leq \delta} |u^{(\nu)}(x)| + C_{11} \right\} t^{(-m-\epsilon)/2}$$

$$0 \leq \nu \leq m+1$$

for $t \geq \delta$.

Furthermore, for $i < m+\epsilon$,

$$(1.3.2) \quad |u^{(i)}(t)| \leq C_{10}(a, \delta, m+\epsilon) \left\{ \max_{0 \leq x \leq \delta} |u^{(\nu)}(x)| + C_{11} \right\} t^{(-m-\epsilon+i)/2}$$

$$0 \leq \nu \leq m+1$$

for $t \geq \delta$.

Also, if $\epsilon = 0$,

$$(1.3.3) \quad |u^{(m)}(t)| \leq C_{13}(a, \delta, m) \left[\max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)| \log\left(2 + \frac{at}{2\pi}\right) \right]$$

for $t \geq \delta$.

Proof: The non-integral term in (1.2.2) is

$$\leq \sum_{r=1}^{\infty} \frac{C_{13}(a, \delta)}{r^{m+2}} \left[\max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)| \right] \exp(-Ct/r^2).$$

By Titchmarsh [11, page 126,

$$\left| \int_0^{\delta} u^{(m+1)}(x) \exp(-s_r x) dx \right| \leq \frac{C_{17}(a, \delta) C_1}{r^{\epsilon}}.$$

Thus, integral term in (1.2.2) is $\leq \sum_{r=1}^{\infty} \frac{C_{13}(a, \delta) C_1}{r^{m+1+\epsilon}} \exp(-Ct/r^2)$.

Hence, (1.3.1) follows from Lemma 1.1.3.

Differentiate (1.2.2) and repeat above process to obtain (1.3.2).

Next we derive (1.3.3).

$$|u^{(m)}(t)| = \left| \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \left[\frac{-a u^{(m)}(\delta)}{s_r} + \int_0^{\delta} u^{(m+1)}(x) \exp(-s_r x) dx \right] \right|$$

(by differentiating (1.2.2))

$$= \left| - \sum_{s_r} \frac{\exp(s_r t) a u^{(m)}(\delta)}{\tau'(s_r) s_r} + \sum_{s_r} \int_0^{\delta} \frac{\exp[s_r(t-x)] u^{(m+1)}(x)}{\tau'(s_r)} dx \right|$$

(by the Lebesgue Dominated Convergence Theorem)

$$\leq C_{19}(a, \delta) u^{(m)}(\delta) t^{-1/2} + \int_0^\delta C_{19}(a, \delta) \log\left(2 + \frac{at}{2\pi}\right) |u^{(m+1)}(x)| dx$$

(by Lemmas 1.1.6 and 1.1.7)

$$\leq C_{19}(a, \delta, m) \left[\max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)| \right] \log\left(2 + \frac{at}{2\pi}\right).$$

We have just shown that sufficiently smooth solutions of (1.2.1) decay at least as fast as some negative power of t . It is also true that no solution of (1.2.1), except $u \equiv 0$, can decay as fast as $\exp(-\frac{a}{\delta} + 1)t)$ as $t \rightarrow +\infty$. This result comes from applying a result due to Wright [13] to our equation.

4. Smoothness and Decay Continued

Theorem 1.3.1 informs us that if $u^{(m+1)}(t)$ ($m > 3$) is continuous on $0 \leq t \leq \delta$, then the solution $u(t)$ of (1.2.1) decays at least as fast as $t^{-m/2}$. In this section we shall veer in the opposite direction and obtain

THEOREM 1.4.1. There exists a solution $u(t)$ of (1.2.1) such that $u^{(m+1)}(t)$ is continuous on $0 \leq t \leq \delta$, yet $u(t)$ decays no faster than $t^{(-m-1-\mu)/2}$ where μ can be chosen arbitrarily > 3 .

Proof: Let

$$(1.4.1) \quad a = 2\pi^2, \quad \delta = 1$$

and let

$$(1.4.2) \quad u(t) = \sum_{q=1}^{\infty} \delta^{-q(m+1+\mu)} [\exp(s_{\beta^q} t) + \exp(\bar{s}_{\beta^q} t)] .$$

Since by Lemma 1.1.3 $s_{\beta^q} = 2\pi\beta^q i + O(\beta^{-q})$, the process of formally differentiating (1.4.2) term by term p times ($p \leq m+1$) yields a uniformly convergent series. Hence, $u(t)$ has $m+1$ continuous derivatives on $0 \leq t \leq \delta$ and is a solution of (1.3.1) for $t \geq 0$. Thus, all we need demonstrate is

LEMMA 1.4.1. Under the assumption (1.4.1), there exists a sequence of points $\{t_k\} \rightarrow 0$ such that

$$(1.4.3) \quad u(t_k) \geq C_2 t_k^{\frac{-m-1-\mu}{2}}$$

where $u(t)$ is defined by (1.4.2).

Proof (of Lemma 1.4.1): Break (1.4.2) up into

$$(1.4.4) \quad \begin{aligned} u(t) &= \sum_{\beta^q < 2(1+C_1)t^{1/3}} \delta^{-q(m+1+\mu)} [\exp(s_{\beta^q} t) + \exp(\bar{s}_{\beta^q} t)] \\ &\quad + \sum_{\beta^q \geq 2(1+C_1)t^{1/3}} \delta^{-q(m+1+\mu)} [\exp(s_{\beta^q} t) + \exp(\bar{s}_{\beta^q} t)] . \end{aligned}$$

This breakup is motivated by the fact that in the former expression on the right side of (1.1.1) all the exponentials have exponents whose real parts are bounded by $-t^{1/3}$ by virtue of (1.1.7) and hence we have exponential decay for this term which we can thus ignore, leaving only the latter expression. Having thus eliminated the first few terms which oscillate between + and - , we aim for a lower estimate as in Lemma 1.1.6.

Let us now consider formally this latter expression.

$$\begin{aligned}
 \sum_{\delta^q} &\geq 2(1+C)t^{1/3} \cdot 3^{-q(m+1+\mu)} \left(\exp(s_{\delta^q} t) + \exp(\bar{s}_{\delta^q} t) \right) \\
 (1.1.5) \quad &= \sum_{\delta^q} \geq 2(1+C)t^{1/3} \cdot 2 \cdot 3^{-q(m+1+\mu)} \exp(\Re[s_{\delta^q} t]) \cos(\Im[s_{\delta^q} t]) \\
 &\geq \sum_{\delta^q} \geq 2(1+C)t^{1/3} \cdot 2 \cdot 3^{-q(m+1+\mu)} \exp\left(\frac{-\pi^2 t}{3^{2q}}\right) \cos\left(\frac{\pi}{3} + \frac{1}{3}\right)
 \end{aligned}$$

(by choosing $t = 3^j$ and t sufficiently large so that q is sufficiently large so that (1.1.7) and (1.1.3) yield a good approximation for s_r where $r = 3^q$; we also use $\delta = 1$ to $\frac{2\pi r t}{\delta}$ in the cosine expression, we use $a = \pi^2$ and the fact we are summing over $r = \delta^q$ to ignore $\frac{at}{2\pi r}$ for $r \leq t$ and obtain $\frac{at}{2\pi r} \leq \frac{\pi}{3}$ for $r > t$, and we use $\delta^q \geq 2(1+C)t^{1/3}$ to estimate the error term by $\frac{1}{3}$)

$$\begin{aligned}
&= \sum_{q=1}^{\infty} 2 \cdot 3^{-q(m+1+\mu)} \exp\left(\frac{-4\pi^2 t}{3^{2q}}\right) \cos\left(\frac{\pi}{4} + \frac{1}{3}\right) \\
&- \sum_{q=1}^{\infty} 2(1+C_1)t^{1/3} 2 \cdot 3^{-q(m+1+\mu)} \exp\left(\frac{-4\pi^2 t}{3^{2q}}\right) \cos\left(\frac{\pi}{4} + \frac{1}{3}\right).
\end{aligned}$$

Using the technique used in proving Lemma 1.1.6 and choosing t sufficiently large we obtain

$$(1.4.5) \quad \sum_{q=1}^{\infty} 2 \cdot 3^{-q(m+1+\mu)} \exp\left(\frac{-4\pi^2 t}{3^{2q}}\right) \cos\left(\frac{\pi}{4} + \frac{1}{3}\right) \geq C_{21} t^{(-m-1-\mu)/2}.$$

Now, we bound the term at the extreme right of (1.4.5).

$$\begin{aligned}
&\sum_{q=1}^{\infty} 2(1+C_1)t^{1/3} 2 \cdot 3^{-q(m+1+\mu)} \exp\left(\frac{-4\pi^2 t}{3^{2q}}\right) \cos\left(\frac{\pi}{4} + \frac{1}{3}\right) \\
(1.4.6) \quad &\leq \sum_{q=1}^{\infty} 2 \cdot 3^{-q(m+1+\mu)} \exp\left(\frac{-4\pi^2 t}{[2(1+C_1)t^{1/3}]^2}\right) \cos\left(\frac{\pi}{4} + \frac{1}{3}\right) \\
&= C_{22} \exp\left(\frac{-\pi^2 t^{1/3}}{(1+C_1)^2}\right).
\end{aligned}$$

Finally, we shall estimate the first term on the right side of

(1.1.4).

$$\begin{aligned}
&\left| \sum_{q=1}^{\infty} 2(1+C_1)t^{1/3} 3^{-q(m+1+\mu)} [\exp(s_{3^q} t) + \exp(\bar{s}_{3^q} t)] \right| \\
(1.4.7) \quad &\leq \sum_{q=1}^{\infty} 2(1+C_1)t^{1/3} 2 \cdot 3^{-q(m+1+\mu)} \exp(\Re[s_{3^q} t])
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{q=1}^{\infty} 2 \cdot 3^{-q(m+1+\mu)} \exp\left(\frac{-\pi^2 t}{[2(1+C_4)]^{1/3} 2}\right) \\
&= C_{23} \exp\left(\frac{-\pi^2 t^{1/3}}{2(1+C_4)^{1/2}}\right).
\end{aligned}$$

Combining (1.4.2), (1.4.3), (1.4.6), (1.4.7), and (1.4.8), we obtain our lemma.

5. Characteristic Roots with Negative Real Parts and Unboundedness

Employing (1.2.1), rewrite (1.3.1) as

$$(1.5.1) \quad L(u) = u'(t+1) - u'(t) + \pi^2 u(t+1) = 0.$$

Consider the function

$$(1.5.2) \quad w(t) = t^2 \sum_{q=1}^{\infty} 3^{-q(3+\mu)} [\exp(s_{3^q} t) + \exp(\bar{s}_{3^q} t)]$$

where $|\mu| > \mu > 0$. $w(t)$ is three times continuously differentiable.

$w(t)$ is thus a bona fide solution of

$$(1.5.3) \quad L^3(w) = L(L(L(w))) = 0.$$

Since the characteristic roots of (1.5.1) all have negative real parts, so do the characteristic roots of (1.5.3). But according to Lemma 1.4.1, there exists a sequence of points $\{t_n\} \rightarrow \infty$ such that

$$(1.5.4) \quad w(t_k) \geq t_k^2 C_{23} t_k^{(-3-)/2} = C_{23} t_k^{(1-)/2}$$

which $\rightarrow \infty$ as $t_k \rightarrow \infty$.

Chapter II

The Inhomogeneous Case

To treat the inhomogeneous equation

$$(2.0.1) \quad u'(t+\delta) - u'(t) + au(t+\delta) = f(t), \quad a > 0, \delta > 0$$

with initial data given on $0 \leq t \leq \delta$, one is tempted to break this up into the homogeneous problem with the given initial data and the inhomogeneous one with zero initial data. This is not the proper procedure to follow because in so doing we are led to two problems each of which may have less smoothness than the original problem which in turn will mean that we will get slower decay than is true for the original problem. However, following a method analogous to that used by Wright [15] in the treatment of the homogeneous case, we are led to two terms in the expression of the solution, one representing the initial data and the other the inhomogeneous term. The advantage of this procedure over the one previously outlined is that boundary terms which would arise from the two separate problems when we integrate by parts and retard decay cancel each other out when the problem is considered as one. This constitutes Section two. Section three will then contain the main result of this chapter, the relation between smoothness and decay. Section one will just contain some elementary lemmas which will permit us to proceed.

We shall generalize the result of Section three to

$$(2.0.2) \quad \sum_{\nu=0}^n [a_{\nu} u^{(\nu)}(t+\delta) + b_{\nu} u^{(\nu)}(t)] = f(t)$$

where all the characteristic roots lie to the left of the imaginary axis, but are asymptotic thereto, in Section four. In particular, if the solution $u(t)$ is $m+n$ times continuously differentiable and $f(t)$ and a sufficient number of its derivatives decay sufficiently fast, then $u(t)$ decays at least as fast as $t^{(-m-n+1)/\theta}$ where the characteristic roots s_r have their real parts approaching the imaginary axis as $-|r|^{-\theta}$. The proof will be omitted; we shall merely indicate the method.

1. Existence and Uniqueness

Lemma 2.1.1 will give us the basic existence, uniqueness, and smoothness theorem for (2.0.1).

LEMMA 2.1.1. If for (2.0.1) we are given that $f(t)$ has m continuous derivatives for $t \geq 0$ and that $u^{(m+1)}(t)$ is continuous for $0 \leq t \leq \delta$, then there exists a unique solution $u(t)$ for $t \geq 0$ and $u^{(m+1)}(t)$ is continuous for $t \geq 0$.

Proof: Interval by interval as for Lemma 1.1.1.

Now we shall state a lemma which will permit us to use the Laplace transform which is needed to derive the expansion theorem in Section two.

LEMMA 2.1.2. If for (2.0.1) we are given that $f(t)$ is piecewise continuous for $t \geq 0$,

$$(2.1.1) \quad |f(t)| \leq C_{2.1} \exp(C_{2.1} t) \text{ for } t \geq 0,$$

and $u'(t)$ is continuous for $0 \leq t \leq \delta$, then

$$(2.1.2) \quad |u(t)|, |u'(t)| \leq C_{25} \exp(C_{27}t) \text{ for } t \geq 0.$$

Proof: See Wright [13].

2. Solution in Terms of Initial Data and Inhomogeneous Term

THEOREM 2.2.1. If $u'(t)$ is continuous for $0 \leq t \leq \delta$ and $f(t)$ is continuous for $0 \leq t \leq T - \delta$, then the solution of (2.3.1) for $\delta \leq t \leq T$ is

$$(2.2.1) \quad u(t) = \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \left[\frac{-au(\delta)}{s_r} + \int_0^{\delta} u'(x) \exp(-s_r x) dx + \int_0^{t-\delta} f(x) \exp(-s_r x) dx \right].$$

Proof: Although the proof of this theorem is essentially presented in Wright [14], we shall present it briefly for two reasons. The first is that Wright does not treat the inhomogeneous case and the second is that since we are in a special case we can simplify the proof.

Note that the solution $u(t)$ of (2.3.1) can depend on $f(x)$ for $0 \leq x \leq t - \delta$ only. Thus, for any t under consideration we shall assume

$$(2.2.2) \quad f(x) \equiv 0 \text{ for all } x > t - \delta.$$

Hence, the hypotheses of Lemma 2.1.2 are satisfied. Hence, (2.1.2) is satisfied.

Let

$$(2.2.3) \quad \operatorname{Re} \tau(s) = C_{23} > C_{27}.$$

Multiply (2.2.1) by e^{-st} , integrate with respect to t from 0 to ∞ (which is permissible as a result of (2.1.2) and (2.2.3)), integrate by parts, shift, and divide by $\tau(s)$. The result will be

$$\begin{aligned} \int_0^{\infty} u(x) e^{-sx} dx &= \frac{e^{\delta s}(s+a)}{s\tau(s)} \int_0^{\delta} u'(x) e^{-sx} dx \\ &\quad - \frac{au(\delta)}{s\tau(s)} + \frac{u(0)}{s} + \frac{1}{\tau(s)} \int_0^{t-\delta} f(x) e^{-sx} dx. \end{aligned}$$

Since $u(t)$ is continuous and of bounded variation for finite range of t (by Lemma 2.1.1), the Laplace inversion formula (see Widder [12]) yields

$$\begin{aligned} (2.2.4) \quad u(t) &= \frac{1}{2\pi i} \lim_{A \rightarrow \infty} \int_{s=C_{20}-iA}^{C_{23}+iA} \left\{ \frac{e^{\delta s}(s+a)}{s\tau(s)} \int_0^{\delta} u'(x) e^{-sx} dx \right. \\ &\quad \left. - \frac{au(\delta)}{s\tau(s)} + \frac{u(0)}{s} \right. \\ &\quad \left. + \frac{1}{\tau(s)} \int_0^{t-\delta} f(x) e^{-sx} dx \right\} e^{st} ds. \end{aligned}$$

Evaluating (2.2.1) via the theory of residues will constitute the remainder of this proof. Let M be the symbol for a large positive integer. Let $\Gamma_1(M)$ be the straight line $\Re s = C_{23}, \frac{-(2M+1)\pi}{\delta} \leq \Im s \leq \frac{(2M+1)\pi}{\delta}$, $\Gamma_2(M)$ be the straight line $\Re s = \frac{(2M+1)\pi}{\delta}, \frac{-(2M+1)\pi}{\delta} \leq \Im s \leq C_{23}$, $\Gamma_3(M)$ be the straight line $\Re s = \frac{-(2M+1)\pi}{\delta}, \frac{-(2M+1)\pi}{\delta} \leq \Im s \leq \frac{(2M+1)\pi}{\delta}$, and $\Gamma_4(M)$ be the straight line $\Re s = \frac{-(2M+1)\pi}{\delta}, \frac{-(2M+1)\pi}{\delta} \leq \Im s \leq C_{23}$.

$$\text{Let } S(s, t) = \left[\frac{e^{\delta s}(s+a)}{s\tau(s)} \int_0^\delta u'(x)e^{-sx}dx - \frac{au(\delta)}{s\tau(s)} + \frac{u(0)}{s} + \frac{1}{\tau(s)} \int_0^{t-\delta} f(x)e^{-sx}dx \right] e^{st}.$$

$$\text{Then } u(t) = \frac{1}{2\pi i} \lim_{M \rightarrow \infty} \int_{\Gamma_1(M)} S(s, t) ds$$

$$= -\frac{1}{2\pi i} \lim_{M \rightarrow \infty} \left[\int_{\Gamma_2(M)} S(s, t) ds + \int_{\Gamma_3(M)} S(s, t) ds + \int_{\Gamma_4(M)} S(s, t) ds \right] \\ + \frac{1}{2\pi i} \sum_{s_r \in C} \oint_{C_r} S(s, t) ds.$$

Elementary, but tedious, computation will show that

$$\lim_{M \rightarrow \infty} \int_{\Gamma_1(M)} S(s, t) ds = 0, \oint_{C_r} S(s, t) ds = 0, \text{ and } \frac{1}{2\pi i} \sum_{s_r \in C} \oint_{C_r} S(s, t) ds \text{ yields}$$

the desired sum.

THEOREM 2.2.2. If $u^{(m+i)}(t)$ is continuous for $0 \leq t \leq \delta$ and $f^{(m)}(t)$ is continuous for $0 \leq t \leq T - \delta$, then the solution of (2.0.1) for $\delta \leq t \leq T$ is

$$\begin{aligned}
 u(t) = & \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \left[\frac{-au^{(m)}(\delta)}{s_r^{m+1}} + \frac{1}{s_r^m} \int_0^\delta u^{(m+1)}(x) \exp(-s_r x) dx \right. \\
 (2.2.5) \quad & \left. + \frac{1}{s_r^m} \int_0^{t-\delta} f^{(m)}(x) \exp(-s_r x) dx \right] \\
 & - \sum_{s_r} \frac{\exp(\delta s_r)}{\tau'(s_r)} \left[\frac{f(t-\delta)}{s_r} + \dots + \frac{f^{(m-1)}(t-\delta)}{s_r^m} \right].
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 u'(t) = & \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \left[\frac{-au^{(m)}(\delta)}{s_r^m} + \frac{1}{s_r^{m-1}} \int_0^\delta u^{(m+1)}(x) \exp(-s_r x) dx \right. \\
 & \left. + \frac{1}{s_r^{m-1}} \int_0^{t-\delta} f^{(m)}(x) \exp(-s_r x) dx \right] \\
 & - \sum_{s_r} \frac{\exp(\delta s_r)}{\tau'(s_r)} \left[\frac{f'(t-\delta)}{s_r} + \dots + \frac{f^{(m-1)}(t-\delta)}{s_r^{m-1}} \right], \\
 (2.2.6) \quad & \dots\dots\dots,
 \end{aligned}$$

$$\begin{aligned}
 u^{(m)}(t) = & \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \left[\frac{-au^{(m)}(\delta)}{s_r} + \int_0^\delta u^{(m+1)}(x) \exp(-s_r x) dx \right. \\
 & \left. + \int_0^{t-\delta} f^{(m)}(x) \exp(-s_r x) dx \right].
 \end{aligned}$$

Proof: (2.2.5) comes from (2.2.1) by integration by parts as in proof of Theorem 1.2.2.

(2.2.6) can be obtained either by differentiating (2.2.5) or by

differentiating (2.0.1) and treating the new problem in similar fashion to the old one.

3. Smoothness and Decay

THEOREM 2.3.1. Let m be a nonnegative integer, $\beta > 1$, and $C_{29} > 0$. If $u^{(m+1)}(t)$ is continuous for $0 \leq t \leq \delta$, $f^{(m)}(t)$ is continuous for $0 \leq t \leq T-\delta$, $u(t)$ is a solution of (2.0.1), and

$$(2.3.1) \quad |f^{(i)}(t)| \leq C_{29}(t+\delta)^{-\beta} \text{ for } 0 \leq t \leq T-\delta, m \geq i \geq 0,$$

then

$$|u^{(i)}(t)| \leq C_{30} \left\{ [C_{29} + \max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)|] t^{(i-m)/2} + C_{29} t^{-\beta+1} \right\}$$

$$\text{for } \delta \leq t \leq T, m-1 \geq i \geq 0,$$

(2.3.2)

$$|u^{(m)}(t)| \leq C_{30} [C_{29} + \max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)|] \log(2 + \frac{at}{2\pi})$$

$$\text{for } \delta \leq t \leq T.$$

C_{30} depends on a, δ, β , and m .

Proof: Our procedure will be to estimate the expressions in

(2.2.5) and (2.2.6).

First consider the terms in the expression for $u(t)$.

$$(2.3.3) \quad \left| \sum_{s_r} \frac{\exp(\delta s_r)}{\tau'(s_r)} \left[\frac{f(t-\delta)}{s_r} + \dots + \frac{f^{(m-1)}(t-\delta)}{s_r^{m-1}} \right] \right| \leq C_{31}(a, \delta, m) C_{29} t^{-\beta}$$

by virtue of (2.3.1).

$$(2.3.4) \quad \left| \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \left[\frac{-au^{(m)}(\delta)}{s_r^{m+1}} + \frac{1}{s_r^m} \int_0^\delta u^{(m+1)}(x) \exp(-s_r x) dx \right] \right|$$

$$\leq C_{32}(a, \delta, m) \max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)| t^{-m/2}$$

by virtue of (1.1.19).

$$(2.3.5) \quad \begin{aligned} & \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \cdot \frac{1}{s_r^m} \int_0^{t-\delta} f^{(m)}(x) \exp(-s_r x) dx \\ &= \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \cdot \frac{1}{s_r^m} \int_0^{(t-\delta)/2} f^{(m)}(x) \exp(-s_r x) dx \\ &+ \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \cdot \frac{1}{s_r^m} \int_{(t-\delta)/2}^{t-\delta} f^{(m)}(x) \exp(-s_r x) dx \end{aligned}$$

$$(2.3.6) \quad \left| \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \cdot \frac{1}{s_r^m} \int_0^{(t-\delta)/2} f^{(m)}(x) \exp(-s_r x) dx \right|$$

$$\leq C_{33}(a, \delta, \beta, m) C_{29} t^{-m/2}$$

by applying (2.3.1) and (1.1.19). And finally,

$$(2.3.7) \quad \left| \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \cdot \frac{1}{s_r} \int_{(t-\delta)/2}^{t-\delta} f^{(m)}(x) \exp(-s_r x) dx \right|$$

$$\leq C_{3-}(a, \delta, \beta, m) C_{29} t^{-\beta+1}$$

by virtue of (2.3.1). Estimating (2.2.5) with the aid of (2.3.3), (2.3.4), (2.3.5), (2.3.6), and (2.3.7), we obtain the proper estimate for $u(t)$.

Similar procedures yield the desired results for all $u^{(i)}(t)$, $i \leq m-1$.

Now, we consider the case of $u^{(m)}(t)$. By the Lebesgue Dominated Convergence Theorem, the expression for $u^{(m)}(t)$ in (2.2.6) becomes

$$(2.3.8) \quad u^{(m)}(t) = - \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \cdot \frac{au^{(m)}(\delta)}{s_r}$$

$$+ \int_0^\delta \sum_{s_r} \frac{\exp(s_r [t-x])}{\tau'(s_r)} u^{(m+1)}(x) dx + \int_0^{t-\delta} \sum_{s_r} \frac{\exp(s_r [t-x])}{\tau'(s_r)} f^{(m)}(x) dx.$$

$$(2.3.9) \quad \left| \sum_{s_r} \frac{\exp(s_r t)}{\tau'(s_r)} \cdot \frac{au^{(m)}(\delta)}{s_r} \right| \leq C_{35}(a, \delta) \max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)| t^{-1/2}$$

by (1.1.19).

$$(2.3.10) \quad \left| \int_0^\delta \sum_{s_r} \frac{\exp(s_r [t-x])}{\tau'(s_r)} u^{(m+1)}(x) dx \right| \leq \delta C_{15}(a, \delta) \max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)| \log \left(2 + \frac{at}{2\pi} \right)$$

by (1.1.23).

$$(2.3.11) \quad \left| \int_0^{t-\delta} \sum_{s_r} \frac{\exp(s_r(t-x))}{\tau'(s_r)} f^{(m)}(x) dx \right| \leq C_{33}(a, \delta, \beta) C_{29} \log(2 + \frac{at}{2\pi})$$

by (2.3.1) and (1.1.23). Applying (2.3.9), (2.3.10), and (2.3.11) to (2.3.3), we obtain the desired estimate for $u^{(m)}(t)$.

4. The General Inhomogeneous Case with One Difference

THEOREM 2.4.1. If $u^{(m+n)}(t)$ is continuous for $0 \leq t \leq \delta$, $m \geq 0$, $f^{(m+n-1)}(t)$ is continuous for $0 \leq t \leq T - \delta$, and

$$(2.4.1) \quad |f^{(i)}(t)| \leq C_{37}(t+\delta)^{-\beta} \text{ for } 0 \leq t \leq T - \delta, m+n-1 \geq i \geq 0,$$

and some $\beta > k$

where k is the greatest multiplicity of any root of the characteristic equation corresponding to (2.3.2),

$$(2.4.2) \quad \sum_{\nu=0}^n [a_{\nu} e^{\delta s} + b_{\nu} s^{\nu}] = 0,$$

then there exists a unique solution $u(t)$ of (2.3.2) and furthermore,

$$(2.4.3) \quad |u^{(i)}(t)| \leq C_{33} \left\{ [C_{37} + \max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+n}} |u^{(\nu)}(x)|] t^{(-n-m+1+i)/\theta} + C_{37} t^{-\beta+k} \right\}$$

for $\delta \leq t \leq T$, $m+n-2 \geq i \geq 0$,

$$|u^{(m+n-1)}(t)| \leq C_{33} [C_{37} + \max_{\substack{0 \leq x \leq \\ 0 \leq \nu \leq m+n}} |u^{(\nu)}(x)| \log(2 + C_{39}t)]$$

for $\delta \leq t \leq T$.

Here C_{33} depends upon $a_\nu, b_\nu, \delta, \beta, m$ and n and θ is determined by

$$(2.4.4) \quad \rho_{\mathcal{L}'}(s_r) \sim -|r|^{-\theta}$$

where s_r are the roots of (2.4.2).

Method of Proof: If (2.4.2) has no multiple roots, the proof is the same as for the special case (2.3.1). Note that it can be shown that (2.4.2) has only finitely many multiple roots and that these have finite multiplicities. Then the solution can be broken up into the contribution of the simple roots which was already treated and the contribution of the multiple roots which can then be estimated.

Chapter III

The Nonlinear Case

Presently, we are in a position to discuss the differential-difference equation

$$(3.0.1) \quad u'(t+\delta) - u'(t) + au(t+\delta) = g(t, u(t), u(t+\delta)), \quad a > 0, \delta > 0$$

with initial data given on $0 \leq t \leq \delta$. Subjecting $g(t, u, v)$ and the initial data to smallness and smoothness conditions, we are able to demonstrate the existence of a unique solution of (3.0.1). This solution has a rate of decay depending on its smoothness which depends upon the smoothness of $g(t, u, v)$ and the initial data.

Regarding $u'(t)$ for $0 \leq t \leq T$ as history and hence known, we can attempt to use the Cauchy-Peano theory for ordinary differential equations to obtain existence and uniqueness for $t > T$. But it is well known that in general this will only be valid in the small. We shall manage, however, to obtain an interval of validity of the order $n^{-1/\epsilon}$ for $n\delta \leq T < (n+1)\delta$ so that we can proceed interval by interval and have our conclusion for all $t \geq 0$.

Having shown existence and uniqueness, our method of proof immediately yields a weak estimate for $u(t+\delta)$ where $t+\delta$ lies in our interval. From this estimate and the conditions imposed upon $g(t, u, v)$, we obtain an estimate for $g(t, u(t), u(t+\delta))$. We are primarily

interested in $u(t+\delta)$, but our method of proof is dependent upon our having results for its derivatives also. So we differentiate (3.0.1) m times. From (3.0.1) itself, we obtain a weak estimate for $u'(t+\delta)$. This allows us to estimate $g^{(1)}(t, u(t), u(t+\delta))$ where for this entire chapter $g^{(i)}(t, u(t), u(t+\delta))$ means the i^{th} total derivative of $g(t, u(t), u(t+\delta))$ with respect to t . Then, from the first differentiation of (3.0.1), we obtain a weak estimate of $u''(t+\delta)$. We keep on going in this fashion until we obtain a weak estimate for $u^{(m)}(t+\delta)$ from the $(m-1)^{\text{st}}$ differentiation of (3.0.1), we estimate $g^{(m)}(t, u(t), u(t+\delta))$, and we read off continuity of $u^{(m+1)}(t+\delta)$ from the m^{th} differentiation of (3.0.1). Having our results for $g(t, u(t), u(t+\delta))$ and its derivatives, we apply Theorem 2.3.1 to (3.0.1) and certain differentiations thereof to obtain our desired result. Our work accomplished, we move on to the next interval. This constitutes Theorem 3.1.1 and its proof.

Theorem 3.1.2 allows a slightly more general right hand side than does Theorem 3.1.1; it suffices for $g(t, u, v)$ and its derivatives to be sufficiently small in t, u , and v rather than in u and v only. It is this form of the theorem we apply in the appendix. We omit the proof as it is the same as for Theorem 3.1.1.

1. Smoothness and Decay

THEOREM 3.1.1. Suppose that $u^{(m+1)}(t)$ is continuous for $0 \leq t \leq \delta$,

$$(3.1.1) \quad \max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)| \leq \epsilon$$

where ϵ is chosen sufficiently small,

$$(3.1.2) \quad m \geq 3,$$

$$(3.1.3) \quad g(t, 0, 0) = g_u(t, 0, 0) = g_v(t, 0, 0) = 0 \text{ for } t \geq 0,$$

and for $\ell+j+k \leq m$, $t \geq 0$, some $C_{\ell j k} > 0$, and

$$(3.1.4) \quad |u|, |v| \leq \epsilon^{3/4} (t+\delta)^{-m/4},$$

we have $\frac{\partial^{(\ell+j+k)} g(t, u, v)}{(\partial t)^\ell (\partial u)^j (\partial v)^k}$ is continuous and

$$(3.1.5) \quad \left| \frac{\partial^{(\ell+j+k)} g(t, u, v)}{(\partial t)^\ell (\partial u)^j (\partial v)^k} \right| \leq C_{\ell j k} (t+\delta)^{2/4}.$$

Then there exists a unique solution of (3.0.1) for all $t \geq 0$

and furthermore,

$$(3.1.6) \quad \begin{aligned} |u^{(i)}(t)| &\leq C_{i,1} \epsilon t^{(i-m)/4} \text{ for } t \geq \delta, m-1 \geq i \geq 0, \\ |u^{(m)}(t)| &\leq C_{i,1} \epsilon \log(2 + \frac{at}{2\pi}) \text{ for } t \geq \delta, \end{aligned}$$

and $u^{(n+1)}(t)$ is continuous for $t \geq 0$. Here C_{n1} depends on a, δ , and m .

Proof: We shall suppose that we have shown the conclusion of our theorem to be true for $t \leq T$ where $n\delta \leq T < (n+1)\delta, n \geq 1$. For simplicity we shall assume $n > 1$. It will then suffice to show it is true for $t \leq T_1 = T + \min(\delta, \frac{1}{2a(n-1)^{1/4}})$ because we can then proceed to show it is true for $t \leq T_1 + \min(\delta, \frac{1}{2aN^{1/4}})$ where $N = n-1$ for $T_1 < (n+1)\delta$ and $N = n$ for $T_1 \geq (n+1)\delta$ and hence we can get as far out as desired in this manner.

First we show existence of $u(t+\delta)$ on $T \leq t+\delta \leq T_1$. Rewrite (3.1.1) as

$$(3.1.7) \quad u'(t+\delta) = u'(t) - au(t+\delta) + g(t, u(t), u(t+\delta)).$$

Choose ϵ sufficiently small so that

$$(3.1.8) \quad \epsilon^{-1/4} \geq 2 \cdot 4^{m/4} [C_{41} + (2a)^{-1} C_{41} \delta^{1/2} + a^{-1} C_{41} (3\delta)^{-m/4} + 1].$$

For $T \leq t+\delta \leq T+\delta$, we have $T-\delta \leq t \leq T$ and by assumption $u'(t)$ is a known function of t , continuous, and bounded by $C_{41} \epsilon^{[(n-1)\delta]^{-(1-m)/4}}$ by (3.1.6). Also, for the rectangle $T \leq t+\delta \leq T+\delta$, $|u| \leq \epsilon^{3/4} [(n+2)\delta]^{-m/4}$, the right hand side of (3.1.7) is continuous. Furthermore, by (3.1.6), $|u(T)| \leq C_{41} \epsilon^{(n\delta)^{-m/4}} < \epsilon^{3/4} [(n+2)\delta]^{-m/4}$ by choice of ϵ . Hence,

we can apply the Cauchy-Peano theorem for ordinary differential equations. (This theorem states: Let $h(x, y)$ be continuous in

$0 \leq x - X \leq b_1, |y - Y| \leq b_2$. Let $M = \max_{\substack{0 \leq x - X \leq b_1 \\ |y - Y| \leq b_2}} |h(x, y)|$. Then there

exists a continuously differentiable solution of $y' = h(x, y)$ for

$0 \leq x - X \leq \min(b_1, \frac{b_2}{M})$ satisfying $y(X) = Y$. Furthermore, if

$|h(x, y_1) - h(x, y_2)| \leq K|y_1 - y_2|$, the solution is unique.) In our case

$x = t + \delta, y = u, h(x, y) = \text{right hand side of (3.1.7)}, X = T, b_1 = \delta,$

$Y = u(T), |Y| \leq C_{41} \epsilon (n\delta)^{-m/4}, b_2 \geq \epsilon^{3/4} [(n+2)\delta]^{-m/4} - C_{41} \epsilon (n\delta)^{-m/4},$

and $M \leq C_{41} \epsilon [(n-1)\delta]^{(-m+1)/4} + a \epsilon^{3/4} [(n+2)\delta]^{-m/4} + 2C_{40} \epsilon^{3/2} [(n+2)\delta]^{-m/2}.$

(To estimate m , we used $|g(t, u, v)| \leq C_{40}(|u|^2 + |v|^2)$. This and

$|g_u(t, u, v)|, |g_v(t, u, v)| \leq C_{40}(|u| + |v|)$ follow from expansion of $g, g_u,$

and g_v in Taylor expansions about $(t, 0, 0)$ and then applying (3.1.3) and

(3.1.5).) Thus, there exists a solution $u(t+\delta)$ of (3.1.7) with $u'(t+\delta)$

continuous in $0 \leq (t+\delta) - T \leq \min(\delta, \frac{b_2}{M})$. A cursory look at the estimates

for b_2 and M suggests $\frac{b_2}{M} \sim (n-1)^{-1/4}$. Actually, by virtue of our

choice of ϵ , we have existence in $T \leq t + \delta \leq T_1 = T + \min(\delta, \frac{1}{2a(n-1)^{1/4}})$.

Indeed, this was the primary reason for the choice of ϵ in (3.1.3).

Next, we demonstrate uniqueness of $u(t+\delta)$ on $T \leq t + \delta \leq T_1$.

Note that in our interval, $u(t)$ is known. Thus, for our uniqueness

theorem, we vary only $u(t+\delta)$ to test for Lipschitz continuity, i.e.

let $u(t), v(t)$ be solutions which agree for $t \leq T$; then

$|[u'(t) - au(t+\delta) + g(t, u(t), u(t+\delta))] - [u'(t) - av(t+\delta) + g(t, u(t), v(t+\delta))]|$

$\leq (a + C_{40})|u - v|$ by (3.1.5). Hence, we have uniqueness.

Now, we show the existence and continuity of $u^{(m+1)}(t+\delta)$ on $T \leq t+\delta \leq T_1$. Differentiation of (3.1.7) m times yields the system

$$\begin{aligned}
 u'(t+\delta) &= u'(t) - au(t+\delta) + g(t, u, v), \\
 u''(t+\delta) &= u''(t) - au'(t+\delta) + g_t(t, u, v) + g_u(t, u, v)u' + g_v(t, u, v)v', \\
 &\vdots \\
 &\vdots \\
 (3.1.9) \quad u^{(m+1)}(t+\delta) &= u^{(m+1)}(t) - au^{(m)}(t+\delta) \\
 &\quad + \sum_{\substack{l+j+k \leq m \\ \sum a_\xi = j \\ \sum b_\eta = k \\ l + \sum b_\eta a_\xi + \sum \eta b_\eta = m}} \alpha_{l, a_1 \dots a_m, b_1 \dots b_m} \left[\frac{\partial^{(l+j+k)} g(t, u, v)}{(\partial t)^l (\partial u)^j (\partial v)^k} \right] \\
 &\quad \cdot (u')^l \cdot \dots \cdot (u^{(m)})^{a_m} (v')^{b_1} \cdot \dots \cdot (v^{(m)})^{b_m}
 \end{aligned}$$

where $u = u(t)$, $v = u(t+\delta)$.

The second relationship of (3.1.9) tells us $u''(t+\delta)$ exists and is continuous on our interval. Etcetera.

Now, we wish to estimate $g(t, u(t), u(t+\delta))$ and its derivatives.

Specifically, we desire

$$\begin{aligned}
 |g^{(i)}(t, u(t), u(t+\delta))| &\leq C_{42} \epsilon^{3/2}(t+\delta)^{(-2m+i)/4} \text{ for } \delta \leq t+\delta \leq T_1, m-1 \geq i \geq 0, \\
 (3.1.10) \quad |g^{(m)}(t, u(t), u(t+\delta))| &\leq C_{42} \epsilon^{3/2}(t+\delta)^{-m/4} \log(2 + \frac{a(t+\delta)}{2\pi}) \text{ for } \delta \leq t+\delta \leq T_1.
 \end{aligned}$$

It will suffice to show this result for $T \leq t+\delta \leq T_1$ because we assume we used (3.1.10) for $\delta \leq t+\delta \leq T$ to obtain (3.1.9) for that interval. The method we shall use to obtain (3.1.10) is to use the properties of $g(t, u, v)$ given by (3.1.3) and (3.1.5) together with

$$(3.1.11) \quad \begin{aligned} |u^{(i)}(t+\delta)| &\leq C_{43} \epsilon^{3/4} [(n+2)\delta]^{(i-m)/4} \text{ for } T \leq t+\delta \leq T_1, m-1 \geq i \geq 0, \\ |u^{(m)}(t+\delta)| &\leq C_{43} \epsilon^{3/4} \log(2 + \frac{an\delta}{2\pi}) \text{ for } T \leq t+\delta \leq T_1. \end{aligned}$$

As was indicated in the introductory remarks of this chapter, to estimate $g^{(j)}(t, u(t), u(t+\delta))$ we shall need (3.1.11) for all $i \leq j$ and to estimate $u^{(j)}(t+\delta)$ by (3.1.11) we shall need (3.1.10) for $i = j-1$.

In our existence proof, we assumed we already had

$|u(T)| \leq C_{41} \epsilon (n\delta)^{-m/4}$ and hence we were permitted to restrict $u(t+\delta)$ by $|u(t+\delta)| \leq \epsilon^{3/4} [(n+2)\delta]^{-m/4}$ for $T \leq t+\delta \leq T_1$. Thus, our proof of (3.1.10) and (3.1.11) gets off the ground. Without loss of generality, assume that these two estimates are proven for all $i < m$ and we desire to verify them for $i = m$.

First, we must estimate $u^{(m)}(t+\delta)$ for $T \leq t+\delta \leq T_1$. For this purpose we shall use the next to the last equation of (3.1.9). By (3.1.6) we have

$$|u^{(m)}(t)| \leq C_{41} \epsilon \log(2 + \frac{a(n+1)\delta}{2\pi}) \leq C_{41} \epsilon^{3/4} \log(2 + \frac{an\delta}{2\pi}).$$

By our assumption for lower derivatives of $u(t+\delta)$,

$$|au^{(m-1)}(t+\delta)| \leq a C_{43} \epsilon^{3/4} [(n+2)\delta]^{-1/4} \leq a C_{43} \epsilon^{3/4} C(\delta) \log(2 + \frac{an\delta}{2\pi}).$$

The remaining term on the right hand side of our equation is

$g^{(m-1)}(t, u(t), u(t+\delta))$ which by (3.1.9) is estimated by

$$|g^{(m-1)}(t, u(t), u(t+\delta))| \leq C_{42} \epsilon^{3/2} (t+\delta)^{(-m-1)/4} \leq C_{42} \epsilon^{3/2} C(\delta) \log(2 + \frac{an\delta}{2\pi}).$$

Thus, we have our estimate (3.1.11) for $u^{(m)}(t+\delta)$ where again we call the constant C_{43} .

Now, we estimate $g^{(m)}(t, u(t), u(t+\delta))$, the expression which appears at the extreme right of the last equation of (3.1.9), for $T \leq t + \delta \leq T_1$. As previously noted, from our existence proof we obtain $|v| = |u(t+\delta)| \leq \epsilon^{3/4} (t+\delta)^{-m/4}$. Also, $|u| \leq C_{41} \epsilon t^{-m/4} \leq \epsilon^{3/4} (t+\delta)^{-m/4}$ by (3.1.6) and (3.1.8). Thus, we stay in the range (3.1.4) in which the estimates (3.1.5) were postulated. This, together with (3.1.3), implies

$$\left| \frac{\partial^{(2+j+k)} g(t, u, v)}{(\partial t)^2 (\partial u)^j (\partial v)^k} \right| \leq \begin{cases} C_{40} (|u| + |v|)^{2-j-k} (t+\delta)^{2/4} & \text{for } j+k \leq 2 \\ C_{40} (t+\delta)^{2/4} & \text{for } j+k > 2. \end{cases}$$

Together with (3.1.4), this implies

$$\left| \frac{\partial^{(2+j+k)} g(t, u, v)}{(\partial t)^2 (\partial u)^j (\partial v)^k} \right| \leq \begin{cases} C_{40} [2\epsilon^{3/4} (t+\delta)^{-m/4}]^{2-j-k} (t+\delta)^{2/4} & \text{for } j+k \leq 2 \\ C_{40} (t+\delta)^{2/4} & \text{for } j+k > 2. \end{cases}$$

Using the induction hypothesis (3.1.11) for $i \leq m$ and $T \leq t + \delta \leq T_1$, we find

$$\left| \prod (v^{(\eta)})^{b_{\eta}} \right| \leq (C_{13} \epsilon^{3/4})^{\sum b_{\eta} [(n+2)\delta]} \sum (\eta-m)b_{\eta} / 4 \log(2 + \frac{an\delta}{2\pi}) \text{ since } v^{(m)} \text{ can}$$

appear at most once in this product. Using the same estimates for $u^{(\xi)}$

(which we can by virtue of (3.1.6) and our choice of ϵ), we find

$$\left| \prod (u^{(\xi)})^{a_{\xi}} (v^{(\eta)})^{b_{\eta}} \right| \leq C_{14} \epsilon^{(3/4)\sum (a_{\xi} + b_{\eta})} \sum (\xi-m)a_{\xi} / 4 + \sum (\eta-m)b_{\eta} / 4 \log(2 + \frac{an\delta}{2\pi})$$

Since $\sum (a_{\xi} + b_{\eta}) = j + k$ and $2 + \sum \xi a_{\xi} + \sum \eta b_{\eta} = m$,

$$\left| \prod (u^{(\xi)})^{a_{\xi}} (v^{(\eta)})^{b_{\eta}} \right| \leq C_{14} (\epsilon^{3/4} [(n+2)\delta]^{-m/4})^{j+k} [(n+2)\delta]^{(m-2)/4} \log(2 + \frac{an\delta}{2\pi}).$$

Combining this with the estimates for the partial derivatives of $g(t, u, v)$,

we obtain the last estimate of (3.1.10), namely

$$\begin{aligned} |g^{(m)}(t, u(t), u(t+\delta))| &\leq C_{42} [\epsilon^{3/4} (t+\delta)^{-m/4}]^{\max(2, j+k)} (t+\delta)^{m/4} \log(2 + \frac{at}{2\pi}) \\ &\leq C_{12} \epsilon^{3/2} (t+\delta)^{-m/4} \log(2 + \frac{at}{2\pi}) \end{aligned}$$

by using 2 in place of $\max(2, j+k)$. Thus, we have (3.1.10) with C_{42} depending on a, δ, m, C_{10} , and C_{11} .

Finally, we are in a position to obtain (3.1.6) for $\delta \leq t \leq T_1$.

It is here and only here that we use the machinery developed in the previous chapters as embodied in Theorem 2.3.1. We use this theorem for different functions on the complete interval $0 \leq t \leq T_1$, namely we apply it to $u^{(i)}(t)$ for $i = 0, 1, 3$, and 5 and choose the appropriate m and β in each case. To avoid confusion let us denote by u^* , m^* , and i^* the u , m , and i appearing in Theorem 2.3.1.

$$\text{Using } m^* = \begin{cases} m-4 & \text{for } i = 0 \text{ or } 1 \\ m-5 & \text{for } i = 3 \text{ or } 5, \end{cases}$$

$\beta = \frac{m-i}{4} + 1$, and $f(t) = g^{(i)}(t, u(t), u(t+\delta))$, we shall now apply Theorem 2.3.1. For $u^*(t) = u^{(5)}(t)$, we shall use the estimate obtained in its entirety, namely for $u^{(i)}(t)$, $m \geq i \geq 5$. However, for $u^*(t) = u(t)$, we shall merely use the estimate obtained for $u(t)$; for $u^*(t) = u'(t)$, we shall merely use the estimate obtained for $u'(t)$ and $u''(t)$; and for $u^*(t) = u^{(3)}(t)$, we shall merely use the estimate obtained for $u^{(3)}(t)$ and $u^{(4)}(t)$.

Using the definition of $f(t)$, then using the estimates (3.1.10) for $g^{(i)}(t, u(t), u(t+\delta))$, then using the definition of β , we have

$$\begin{aligned} |f^{(i^*)}(t)| &= |g^{(i^*+i)}(t, u(t), u(t+\delta))| \text{ (for } i = 0, 1, 3, \text{ or } 5) \\ &\leq \begin{cases} C_{42} \epsilon^{3/2} (t+\delta)^{(-2m+i^*+i)/4} & \text{for } \delta \leq t+\delta \leq T_1, m-1 \geq i^*+i \geq 0, m^* \geq i^* \geq 0 \\ C_{42} \epsilon^{3/2} (t+\delta)^{-m/4} \log(2 + \frac{a(t+\delta)}{2\pi}) & \text{for } \delta \leq t+\delta \leq T_1, i = 5, i^* = m-5 \end{cases} \\ &\leq C_{42} \epsilon^{3/2} (t+\delta)^{-\beta}. \end{aligned}$$

Hence, by Theorem 2.3.1 applied for each $u^{(i)}(t)$ ($i = 0, 1, 3, \text{ or } 5$), we have

$$|u^{(i^*+i)}(t)| \leq C_{30} [C_{42} \epsilon^{3/2} + \epsilon] t^{(i^*-m^*)/2} + C_{42} \epsilon^{3/2} t^{-\beta+1}$$

$$\text{for } \delta \leq t \leq T_1, m^* - 1 \geq i^* \geq 0,$$

(3.1.12)

$$|u^{(m^*+i)}(t)| \leq C_{30} [C_{42} \epsilon^{3/2} + \epsilon] \log(2 + \frac{at}{2\pi})$$

for $\delta \leq t \leq T_1$.

For $i = 0$, let $i^* = 0$; for $i = 1$ or 3 , let $i^* = 0$ or 1 ; and for $i = 5$, let i^* be any integer satisfying $0 \leq i^* \leq m-5$. Note that in this way each integer from 0 to m corresponds uniquely to one of our $i^* + i$. We defined $\beta = \frac{m-i}{4} + 1$. Also, from $m \geq 3$, the definition of m^* , and our restriction of i^* , we obtain

$$\frac{m^*-i^*}{2} \geq \frac{m-i^*-i}{4}. \text{ Choose}$$

$$(3.1.13) \quad C_{41} \geq \epsilon C_{30} \text{ and } \epsilon^{-1/2} \geq C_{42}.$$

We now have (3.1.6) for $\delta \leq t \leq T_1$ and can proceed to the next interval.

THEOREM 3.1.2. Suppose that $u^{(m+1)}(t)$ is continuous for

$$0 \leq t \leq \delta,$$

$$(3.1.14) \quad \max_{\substack{0 \leq x \leq \delta \\ 0 \leq \nu \leq m+1}} |u^{(\nu)}(x)| \leq \epsilon$$

where ϵ is chosen sufficiently small,

$$(3.1.15) \quad m \geq 8,$$

and for $\ell + j + k \leq m$, $t \geq 0$, some $C_{15} > 0$, and

$$(3.1.16) \quad |u|, |v| \leq \epsilon^{3/4} (t+\delta)^{-m/4},$$

we have $\frac{\partial^{(\ell+j+k)} g(t, u, v)}{(\partial t)^\ell (\partial u)^j (\partial v)^k}$ is continuous and

$$(3.1.17) \quad \left| \frac{\partial^{(\ell+j+k)} g(t, u, v)}{(\partial t)^\ell (\partial u)^j (\partial v)^k} \right| \leq C_{45} [\epsilon^{3/4} (t+\delta)^{-m/4}]^{2-j-k} (t+\delta)^{\ell/4}.$$

Then there exists a unique solution of (3.0.1) for all $t \geq 0$ and
furthermore,

$$|u^{(i)}(t)| \leq C_{46}(a, \delta, m) \epsilon t^{(i-m)/4} \text{ for } t \geq \delta, m-1 \geq i \geq 0,$$

(3.1.13)

$$|u^{(m)}(t)| \leq C_{46}(a, \delta, m) \epsilon \log \left(2 + \frac{at}{2\pi}\right) \text{ for } t \geq \delta,$$

and $u^{(m+1)}(t)$ is continuous for $t \geq 0$.

Appendix

The Wave Equation with a Nonlinear Interface Condition

Consider a segment of transmission line of length $l_1 + l_2$. Place a generator at the left end of the line and an impedance at the right end of the line. Also, place a capacitor in parallel with the line at a distance l_1 from the generator.

Let $v(x, t)$ denote voltage at point x and time t . Let the capacitance of the capacitor depend upon its voltage and be denoted $c(v)$. Let $\phi(t)$ be the input voltage at the generator. We shall assume that the voltage and the current gradient down the line are initially zero. We further assume that the impedance is matched to the line. Let $[f]$ denote the jump in f with respect to x .

Miranker [9] showed that the above problem leads to the following set of equations for $v(x, t)$.

$$(A.1) \quad v_{xx} = \alpha^2 v_{tt} \text{ for } t > 0, -l_1 < x < l_2, x \neq 0,$$

$$(A.2) \quad v(x, 0) = v_t(x, 0) = 0,$$

$$(A.3) \quad v_x + \alpha v_t = 0 \text{ for } x = l_2,$$

$$(A.4) \quad v = \phi(t) \text{ for } x = -l_1,$$

and

$$(A.5) \quad [v_x] = 2\alpha(c(v)v_t)_t \text{ for } x = 0.$$

It is to be noted that $\alpha > 0$ and that $c(v)$, whereas not the same as $c(v)$ above, is a positive constant multiple thereof.

Then using Gauss' Theorem, he showed that this can be reduced to solving

$$(A.6) \quad c(v)v_t \Big|_{t-\tau}^t = -v(t) + \phi(t - \frac{\tau}{2})$$

along $x = 0$. Here $\tau = 2\alpha l_1$.

Presently, it is our aim to show that if $c(v)$, $v(0, t)$, and $\phi(t)$ are sufficiently smooth, $v(0, t)$ and a proper number of its derivatives on $0 \leq t \leq \tau$ are small, and $\phi(t)$ and a sufficient number of its derivatives decay as some high enough negative power of t , then $v(x, t)$ decays for all x in our region.

The proof of decay for a solution of (A.1)-(A.4) where (A.1) (rather than (A.5)) holds for $x = 0$ is very elementary. Let the solution $v(x, t) = A(t + \alpha x) + B(t - \alpha x)$. Then from (A.3) we obtain $A'(y) = 0$ for $y \geq \alpha l_2$. Whereas the solution of the differential equation is unique, we can subtract a constant from $A(y)$ as long as we add it to $B(y)$. So we subtract $A(\alpha l_2)$ from $A(y)$ and add it to $B(y)$. We thus have $A(\alpha l_2) = 0$. Hence, $v(x, t) = B(t - \alpha x)$ for $t + \alpha x \geq \alpha l_2$. Thus, $v(x, t)$ decays as $\phi(t)$ decays.

Our problem is, however, that the jump condition (A.5) holds at $x = 0$. Converting (A.3) to (3.0.1) and then applying Theorem 3.1.2 thereto, we obtain decay along $x = 0$. Then for the region $x \geq 0$ we have the elementary problem handled above and for the region $x \leq 0$ we shall show that we can obtain decay despite the break along $x = 0$.

First, we convert (A.6) to (3.0.1). Let

$$(A.7) \quad u(t) = \int_0^{v(t)} c(\lambda) d\lambda.$$

Then

$$(A.8) \quad u'(t) = c(v)v'(t).$$

Let $F(u, v) = -u + \int_0^v c(\lambda) d\lambda = 0$. Then $F_v(u, v) = c(v) > 0$ since capacitance is always positive (actually nonnegative, but for our purpose we need positive). Hence, by the implicit function theorem, we can solve $F(u, v) = 0$ and obtain $v = \psi(u)$. Since $c(v) > 0$, $u = 0$ implies $v = 0$ by virtue of (A.7). Also, if we consider u and v as functions of t , $u'(t) = c(0)v'(t)$ at $u(t) = 0$ by virtue of (A.8). Thus, if v is sufficiently smooth, $v = \frac{u}{c(0)} + u^2 g(u)$. Let $\frac{1}{c(0)} = a$. We then have

$$(A.9) \quad u'(t) - u'(t-\tau) + au(t) = -u^2(t)g(u(t)) + \phi\left(t - \frac{\tau}{2}\right)$$

which is equivalent to (3.0.1). However, Theorem 3.1.1 is inapplicable since the right side of (A.9) does not vanish for $u(t) = u(t-\tau) = 0$. Therefore, we shall use Theorem 3.1.2.

If $u(t)$ for $0 \leq t \leq \tau$ and the right hand side of (A.9) for all $t \geq 0$ are sufficiently smooth, $u(t)$ and a proper amount of its derivatives for $0 \leq t \leq \tau$ are small, and $\phi(t)$ and an adequate number of its derivatives decay as some sufficiently high negative power of t , then by Theorem 3.1.2 we have decay for $u(t)$ and several of its derivatives. Now, our conditions on c, v , and ϕ give us these conditions. Thus, we have decay for $u(t)$ and several of its derivatives and hence for $v(0, t)$ and several of its derivatives.

For $x \geq 0$, we have the elementary problem discussed above for the case of no jump condition (A.5). Hence $v(x, t) = B(t - \alpha x)$ for $t + \alpha x \geq \alpha l_2$. Also, $v(0, t)$ is smooth. Thus, $\frac{\partial^{(i)} v(x, t)}{(\partial t)^i} = B^{(i)}(t - \alpha x)$ for as many derivatives as $v(0, t)$ has. Thus, decay of $v(0, t)$ and several of its derivatives carries over to decay of $v(x, t)$ and several of its derivatives for $0 \leq x \leq l_2$.

Note that the preceding paragraph implies $v_x + \alpha v_t = 0$ for $t + \alpha x \geq \alpha l_2, x \geq 0$. Hence, $v_x + \alpha v_t = 0$ along $x = 0$ for $t \geq \alpha l_2$ where by v_x we mean the right handed derivative. Together with (A.5) this implies $v_x = -\alpha v_t - 2\alpha(c(v)v_t)_t$ for $x \leq 0, t \geq \alpha l_2$. Along $x = 0$, $v_t = \frac{u'(t)}{c(0)}$ and $(c(v)v_t)_t = u''(t)$ both decay by Theorem 3.1.2. Thus,

v_t and v_x both decay along the left hand side of $x = 0$. Let

$v(x, t) = D(t + \alpha x) + E(t - \alpha x)$ for $x \leq 0$. Then $D'(t)$ and $E'(t)$ both

decay. If we have sufficient decay for $u'(t)$ and $u''(t)$, we have $D'(t)$

and $E'(t)$ decaying sufficiently so that $D(t) = D_\infty - \int_t^\infty D'(y)dy$ and

$E(t) = E_\infty - \int_t^\infty E'(y)dy$. But we are given $\phi(t) = D(t - \alpha l_1) + E(t + \alpha l_1)$

decays. Thus, $D_\infty - \int_{t-\alpha l_1}^\infty D'(y)dy + E_\infty - \int_{t-\alpha l_1}^\infty E'(y)dy$ decays. Hence,

$D_\infty + E_\infty = 0$, $D_\infty = -E_\infty$. Add D_∞ to $E(t - \alpha x)$ and subtract it from

$D(t + \alpha x)$. Then $D(t + \alpha x)$ and $E(t - \alpha x)$ both decay. So does $v(x, t)$.

Thus, $v(x, t)$ decays for all x in our region.

Bibliography

- [1] Bellman, R., On the Existence and Boundedness of Solutions of Non-Linear Differential-Difference Equations, *Ann. Math.*, 50(1949), pp. 347-355.
- [2] Bellman, R. and Cooke, K. L., *Differential-Difference Equations*, Academic Press, 1963.
- [3] Coddington, E. A. and Levinson, N., Theory of Ordinary Differential Equations, McGraw-Hill, 1955.
- [4] Cooke, K. L., On Transcendental Equations Related to Differential-Difference Equations, *J. Math. Anal. Appl.*, 4(1962), pp. 65-71.
- [5] Hahn, W., Zur Stabilität der Lösungen von Linearen Differential-Differenzengleichungen mit Konstanten Koeffizienten, *Math. Ann.*, 131(1956), pp. 151-156.
- [6] Hahn, W., Über Differential-Differenzengleichungen mit Anomalen Lösungen, *Math. Ann.*, 133(1957), pp. 251-255.
- [7] Hohn, G., Randwertaufgaben und Funktionale Differentialgleichungen, *Jber. Dtsch. Math. Verein.*, 39(1935), pp. 54-58.
- [8] Langer, R. E., On the Zeros of Exponential Sums and Integrals, *Bull. AMS*, 37(1931), pp. 213-239.
- [9] Miranker, P. L., Periodic Solutions of the Wave Equation with a Nonlinear Interface Condition, *IEE J. Res. Dev.*, 5(1931), pp. 2-2.

- [10] Pontryagin, L. S., On the Zeros of Some Transcendental Functions, Izv. Akad. Nauk SSSR. Ser. Mat., 3(1952), pp. 115-131; AMS Translations (2) 1(1955), pp. 95-110.
- [11] Titchmarsh, E. C., The Theory of Functions, second edition, Oxford, 1939.
- [12] Widder, D. V., The Laplace Transform, Princeton, 1946.
- [13] Wright, E. L., Linear Difference-Differential Equations, Proc. Camb. Phil. Soc., 44(1948), pp. 179-185.
- [14] Wright, E. L., The Linear Difference-Differential Equation with Constant Coefficients, Proc. Roy. Soc. Edinburgh, section A, 32(1943-9), pp. 337-393.

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author) Courant Institute of Mathematical Sciences New York University, 25 Waverly Place, New York, New York 10003		2a REPORT SECURITY CLASSIFICATION Non-classified	
		2b GROUP None	
3 REPORT TITLE EXISTENCE, UNIQUENESS, AND STABILITY FOR NONLINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS IN THE NEUTRAL CASE			
4 DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report February 1965			
5 AUTHOR(S) (Last name, first name, initial) Snow, Wolfe			
6. REPORT DATE February 1965		7a. TOTAL NO. OF PAGES 54	7b. NO. OF REFS 14
8a. CONTRACT OR GRANT NO. NONR 285 (55)		9a. ORIGINATOR'S REPORT NUMBER(S) IMM-NYU 328	
b. PROJECT NO. NR 062-160			
c.		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) NONE	
d.			
10 AVAILABILITY/LIMITATION NOTICES Qualified requesters may obtain copies of this report from DDC.			
11. SUPPLEMENTARY NOTES NONE		12 SPONSORING MILITARY ACTIVITY U.S. Navy, Office of Naval Research 207 West 24th Street New York, New York	
13 ABSTRACT For ordinary differential equations it is well known that an equilibrium solution is stable if the linearized equation has only exponentially decaying solutions. The differential-difference equation $u'(t+\sigma) - u'(t) + au(t+\sigma) = g(t, u(t), u(t+\sigma)), a > 0, \sigma > 0,$ when linearized, also has decaying exponentials $\exp(st)$, but the characteristic roots s approach the imaginary axis. The problem arises -- do solutions of this equation decay? An affirmative answer is obtained for this problem, the rate of de- cay depending upon the smoothness of the initial data and nonlinear term and the rate of approach of the characteristic roots to the imaginary axis. Then this result is applied to a transmission line problem. Also, an example is given of a homogeneous linear ordinary differ- ential-difference equation with constant coefficients, whose characteristic roots all lie in the left half plane, having an un- bounded solution.			

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